Mathematics SL

Internal Assessment

What is the relationship between the area inside a parabola and under its curve?

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Introduction

The process of finding a function, given its derivative, is called integration[1]. An an integral assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data[2]. This investigation aims to understand what percentage the integrated area of a given parabola makes of its area. Therefore, my research question is, "*What is the relationship between the area inside a parabola and the area under its curve?"*. I have quite an interest in design; namely architecture. It has always intrigued me and I have understood that the role of Mathematics in it is very crucial. I have seen many fascinating bridges in my life and two of my favourites are the Van Stadens Bridge and the Golden Gate Bridge. They are both Arc bridges and their design is very interesting. They are evidently in the shape of parabolas. Interestingly, I noticed a pattern; the supporting structure is close to half of empty space if I were to draw an imaginary rectangle along the supporting structures. Therefore, through my exploration, I aim to understand whether such parabolic structures tend be in a general proportion to maintain their stability.

My hypothesis is that there is a strong correlation between these two values. For example, in the equation $3x^2$, if a rectangle is plotted on the graph form -3 to 3, the area under the curve within these limits is 54 units^2. The area of the rectangle would be 162 units^2 and the area of the parabola would be 108 units^2. This way, I now know that the area under the curve is 50% of the parabolas area when it is bounded by the rectangle. In the same way, I will try to understand the correlation between the two values and then from a general equation for it.

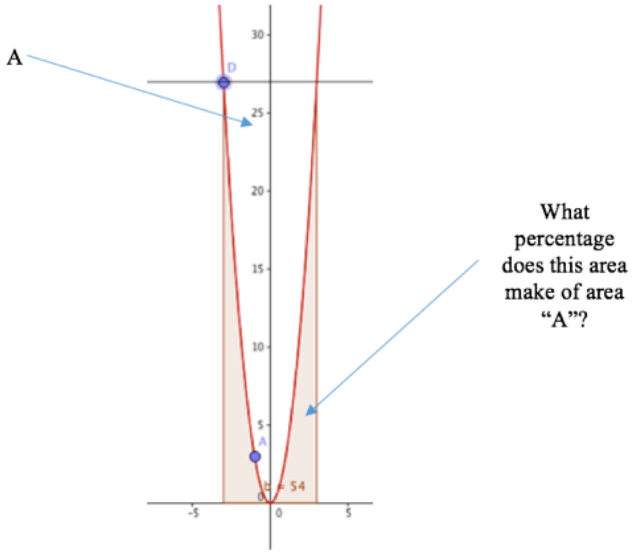


Image 1 – Appendix 1

[1] "Integration." *Math First*. Massey University, 2003. Web. 19 Jan. 2016. <http://mathsfirst.massey.ac.nz/Calculus/integration/basics/definition.html>.

[2] Integral." *Wikipedia*. Wikimedia Foundation, 2012. Web. 18 Jan. 2016. <https://en.wikipedia.org/wiki/Integral>.

For my investigation, I will take a few equations with different co-efficients. For each equation, I will increase the range in order from, $\pm 1 \; to \pm 10$. I will then make the necessary calculations to find the correlation after which I will analyse the actual bridges.

Sample Working of Integration:

$$\int_{-1}^{1} 3x^2 \, dx$$

1) *Solving the in-definite Integral:*

$= 3 \int x^2 \, dx$

$= 3 \frac{x^{2+1}}{2+1} + \mathrm{c}$

$= x^3 + c$

Adding boundaries:

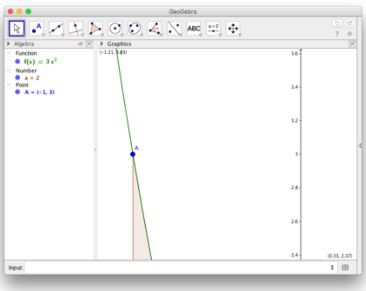


Image 2 – Appendix 1

$= \lim_{x \to -1} (x^3) = -1$

$= \lim_{x \to 1} (x^3) = 1$

Subtracting lower limit from – higher limit:

$= 1 - (-1)$

$= 2$

2) *Finding y coordinate of intersection point using GeoGebra:*

In this, I will find the y coordinate of the intersection point to find the maximum point. This way, I will be able to find the area of the rectangle as I will multiply the range, '2', (-1 to 1) with the maximum point 3, to get the area of 6 units^2

3) *finding the areas:*

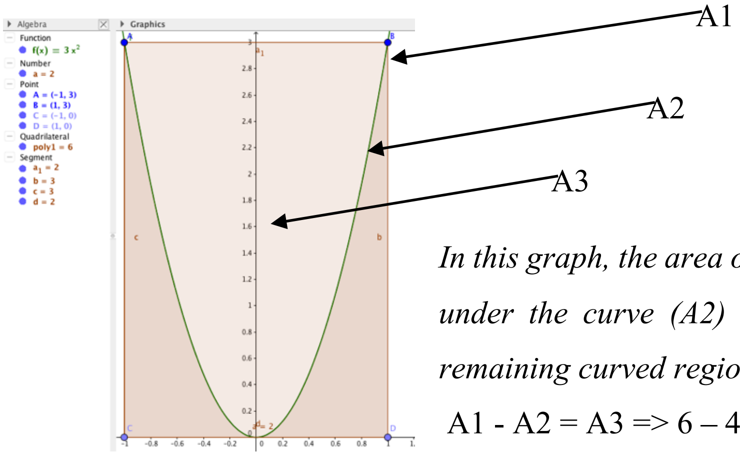


Image 3 – Appendix 1

In this graph, the area of rectangle (A1) ABCD is 6. The area under the curve (A2) is 2 and therefore, the area of the remaining curved region (A3) is 4.

A1 - A2 = A3 => 6 – 4 = 2

4) *Now, my final step would be to find out what percentage does the Integrated area (A2), make of the residual curve area (A3). I solved this using the unitary method:*

$if\ 4 \rightarrow 100$

$then\ 2 \rightarrow ?$

Cross multiplying:

$\frac{2 \times 100}{4} = 50\%$

In the same way, I solved a few sums with different coefficient for varying ranges to see if the pattern emerged for each equation; Here is a raw data table of my working:

Table 1. Raw Data Calculations

Coefficient of x^2	Lower Limit	Upper Limit	Range (units)	A2 [Area under the curve] ($unit^2$)	Max Point	A1 [Total Area] ($unit^2$)	A3 [Curve Area] ($unit^2$)	Percentage %
3	-1	1	2	2	3	6	4	50
3	-2	2	4	16	12	48	32	50
3	-3	3	6	54	27	162	108	50
3	-4	4	8	128	48	384	256	50
3	-5	5	10	250	75	750	500	50
3	-6	6	12	432	108	1296	864	50
3	-7	7	14	686	147	2058	1372	50
3	-8	8	16	1024	192	3072	2048	50
3	-9	9	18	1458	243	4374	2916	50
3	-10	10	20	2000	300	6000	4000	50
4	-1	1	2	2.67	4	8	5.33	50.09
4	-2	2	4	21.33	16	64	42.67	49.98
4	-3	3	6	72	36	216	144	50
4	-4	4	8	170.67	64	512	341.33	50
4	-5	5	10	333.33	100	1000	666.67	49.99
4	-6	6	12	576	144	1728	1152	50
4	-7	7	14	914.67	196	2744	1829.33	50
4	-8	8	16	1365.33	256	4096	2730.67	49.99
4	-9	9	18	1944	323	5814	3870	50.23
4	-10	10	20	2666.67	400	8000	5333.33	50
-5	-1	1	2	3.33	5	10	6.67	49.92
-5	-2	2	4	26.67	20	80	53.33	50
-5	-3	3	6	90	45	270	180	50

-5	-4	4	8	213.33	80	640	426.67	49.99
-5	-5	5	10	416.67	125	1250	833.33	50
-5	-6	6	12	720	180	2160	1440	50
-5	-7	7	14	1143.33	245	3430	2286.67	49.99
-5	-8	8	16	1706.67	320	5120	3413.33	50
-5	-9	9	18	2430	405	7290	4860	50
-5	-10	10	20	3333.33	500	10000	6666.67	49.99
-2	-1	1	2	1.33	2	4	2.67	49.81
-2	-2	2	4	10.67	8	32	21.33	50.02
-2	-3	3	6	36	18	108	72	50
-2	-4	4	8	85.33	32	256	170.67	49.99
-2	-5	5	10	166.67	50	500	333.33	50
-2	-6	6	12	288	72	864	576	50
-2	-7	7	14	457.33	98	1372	914.67	49.99
-2	-8	8	16	682.67	128	2048	1365.33	50
-2	-9	9	18	972	162	2916	1944	50
-2	-10	10	20	1333.33	200	4000	2666.67	49.99

Through the tables, It is quite evident that the value is close to 50% almost every time. So, I have now decided to find the correlation between the coefficient of the equation and the percentage found.

Finding the Pearson Correlation[3]:

I will first find the average of each coefficient's respective percentage.

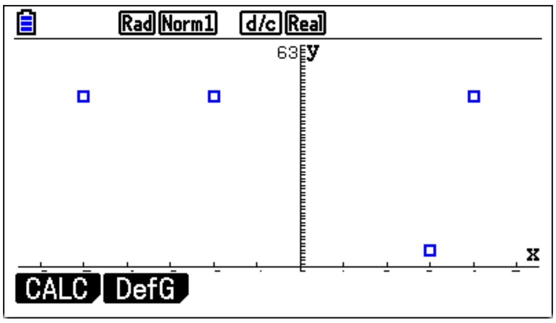
Sample working for $3x^2$; ***Finding the the average (mean) :***

$$\frac{50+50+50+50+50+50+50+50+50+50}{10} = 50$$

1) *Bivariate Analysis; Finding 'r' (Pearson's correlation coefficient)*

Table 2. Variables

Coefficient (x)	Percentage (y)
3	50
4	50.03
-2	49.98
-5	49.99



Graph 1(Scatter Plot) – Table 2

[3] Cirrito, Fabio. *IBID - Mathematics Standard Level*. N.p.: n.p., 2016. Print.

Table 3. Statistic Measurements

	Coefficient (x)	**Average (y)**	xy	x^2	y^2
	3	50	150	9	2500
	4	50.03	200.12	16	2503.0009
	-2	49.98	-99.96	4	2498.0004
	-5	49.99	-249.95	25	2499.0001
Σ	0	200	0.21	54	10000.0014

1) *Substituting Values in 'r' formula*

$$r = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\left(\sqrt{\sum x^2 - \frac{(\sum x)^2}{n}}\right)\left(\sqrt{\sum y^2 - \frac{(\sum y)^2}{n}}\right)}$$

$$r = \frac{0.21 - \frac{(0)(200)}{4}}{\left(\sqrt{54 - \frac{(0)^2}{4}}\right)\left(\sqrt{10000.0014 - \frac{(200)^2}{4}}\right)}$$

$$r = \frac{0.21}{(\sqrt{54})\left(\sqrt{\frac{7}{5000}}\right)}$$

$$r = 0.7637626158$$

Therefore, because r > 0.6, it is a Moderate Positive Linear Correlation.

a = $3.8888E^{-3}$

a = $\frac{3.8888}{1000}$

a = 0.0038888

b = 50

Therefore, equation of regression in the form, y = a(x) + b

y = 0.0038888 (x) + 50

Now that I have found that there is a correlation between the coefficient and the percentage covered, I know that they have a moderate correlation. I will now move on to looking at the two bridges, modelling their equation and finding out whether they are designed as per my predictions or not.

I have used a digital math software known as GeoGebra[4] to model the equation for the bridge. I took images from online sources and tried my best to make sure that they were exact side/perpendicular views of the bridges. I then imported them into the software and marked the exact points along the curved support structures. Then, using the software's tools, I modelled an equation for the curve to find the coefficient. I also used the scaling tool to find the length of the bridges from one end to another in order to find the lower and upper limits while integrating it. The first bridge that I will be modelling an equation for is the Van Stadens Bridge.

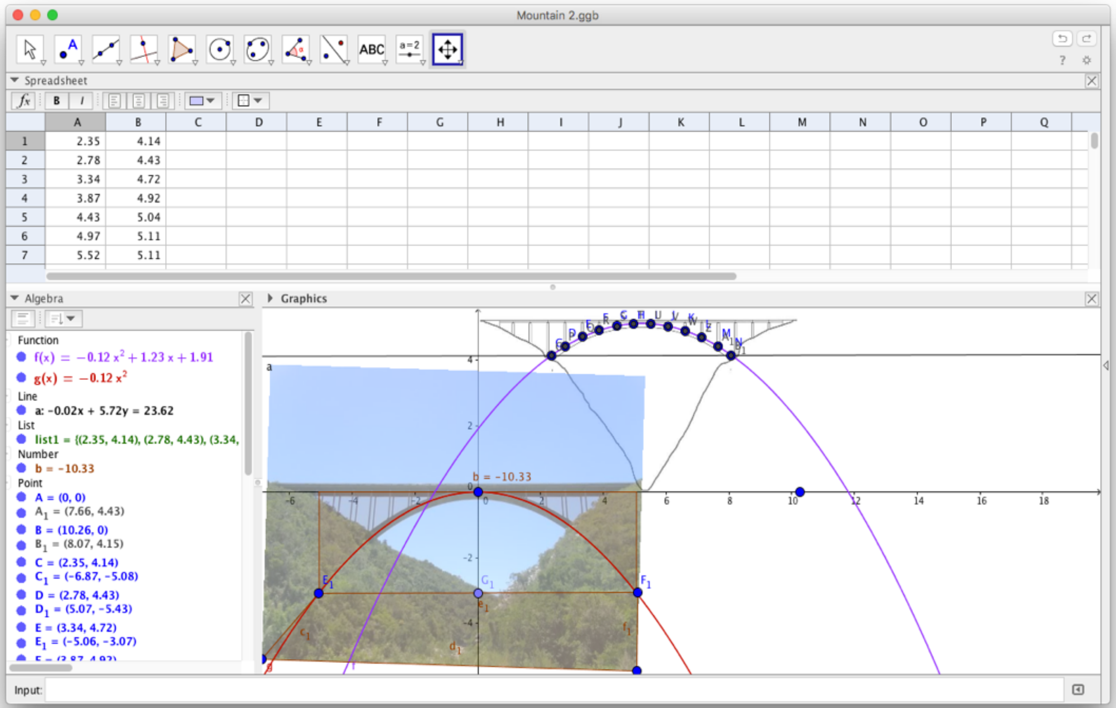


Image 5 – Appendix 1

The equation for this graph is $-0.12x^2 + 1.23x + 1.91$. Once I had the coefficient of x^2, I moved on to integrating the equation for the lower and upper limits that I found; -5.055 to +5.055 respectively. The range of the lower to upper limit is actually the length of the bridge so I first calculated the horizontal length of the bridge and divided it by 2, to find the limits. I also moved the graph down to the origin so the equation becomes just $-0.12x^2$. This was just to make the calculations simpler, nothing else in particular. I calculated the vertex and shifted it to the origin so as to have just the equation in the simple form of $y = ax^2$.

$$\int_{-5.055}^{5.055} -0.12x^2 \, dx$$

[4] *GeoGebra*. Computer software. Vers. 5.0. N.p., n.d. Web. 12 Jan. 2016.

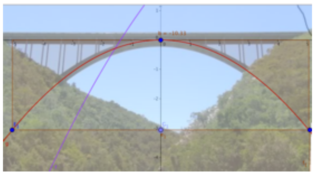
1) Solving the in-definite Integral:

$= -0.12 \int x^2\, dx$

$= -0.12 \frac{x^{2+1}}{2+1} + \mathrm{c}$

$= \frac{-x^3}{25} + c$

2) Adding boundaries:



$= \lim_{x \to -5.055} \left(\frac{-x^3}{25}\right) = 5.16682$

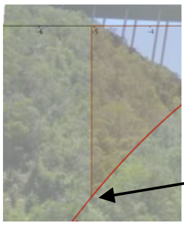
$= \lim_{x \to 5.055} \left(\frac{-x^3}{25}\right) = -5.16682$

Image 6 – Appendix 1

3) Subtracting lower limit from – higher limit:

$= -5.16682 - 5.16682$

= -10.3336



4) Finding Y coordinate of intersecting point:

Using GeoGebra, I was able to find the y coordinate; (0, -3.06) This is the point of intersection between the parabola and the integration boundaries.

Image 7 – Appendix 1

5) Finding Areas:

I now have the integrated area, 10.33. The area of the rectangle bounding the curve is 30.93. It is the product of the range (10.11) (units between the lower and upper limit; -5.055 to 5.055) and the minimum point -3.06.Therefore, the area of the parabola would be the total area – the integrated area (30.9366 – 10.33 = 20.6066).

6) Finding Percentages:

$if\ 20.6066 \rightarrow 100$

$then\ 10.33 \rightarrow ?$

Cross multiplying:

$\frac{10.33 \times 100}{20.6066} = 50.12\%$

Table 4. Calculations for Graph of Bridge 1

(A)x²	From	To	Range	Integrated Area	Min Point	Total Area	Curve Area	Percentage
-0.12	-5.055	5.055	10.11	10.33	3.06	30.9366	20.6066	50.12

From these calculations, it is evident that my predictions have been right for the first bridge. I will now move on to analysing the second bridge.

The second bridge that I will be modeling an equation for is The Golden Gate Bridge. The equation for this graph is $0.13x^2 - 0x + 0.05$. Once I had the coefficient of x^2, I moved on to integrating the equation for the lower and upper limits that I found; -1.81 to +1.82 respectively. This was the horizontal length, that I had split into 2. In this graph, I haven't use the general form of $y = ax^2$ because the minimum point of this parabola is at y = 0.05. This bridge is curved form the bottom as well and I will also keep this in mind while making my calculations.

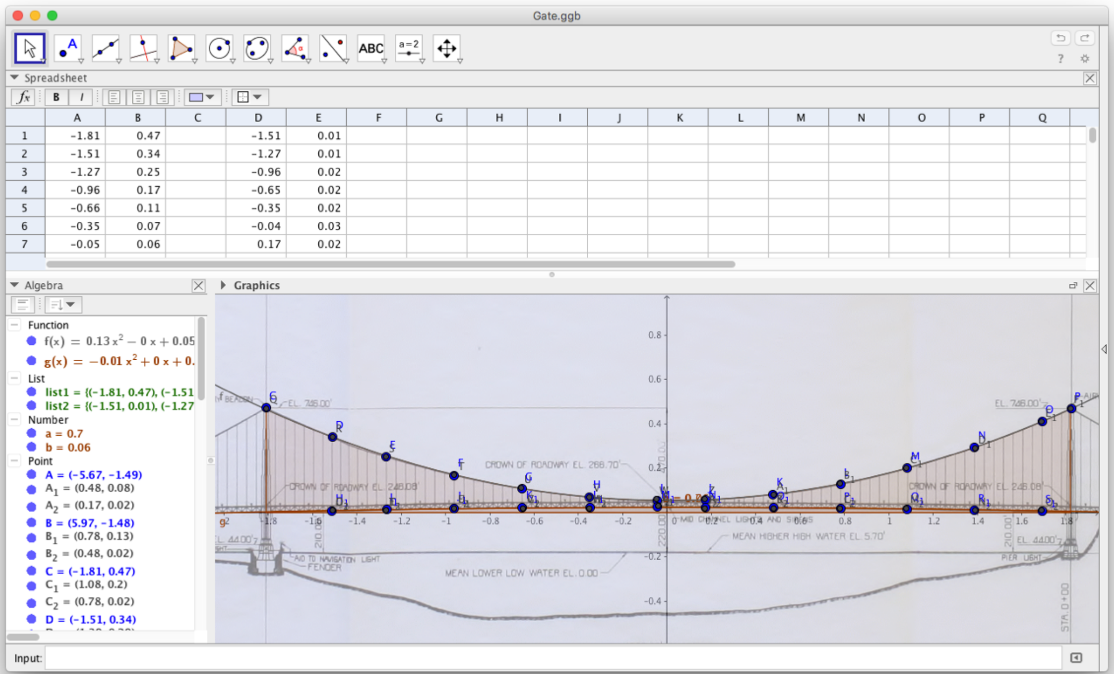


Image 8 – Appendix 1

$$\int_{-1.81}^{1.82} 0.13x^2 + 0.05\, dx$$

1) *Applying sum rule:*

$$= \int 0.13x^2 dx + \int 0.05 dx$$

2) *Solving the in-definite Integral:*

$$= 0.13 \int x^2\, dx \;= 0.13\frac{x^{2+1}}{2+1} \;= \frac{13x^3}{300}$$

$$= 0.05 \int dx = \frac{x}{20}$$

$$= \frac{13x^3}{300} + \frac{x}{20} + c$$

3) *Adding boundaries:*

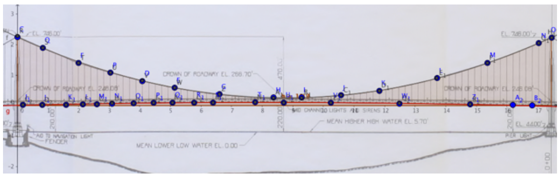


Image 9 – Appendix 1

$$= \lim_{x \to -1.81} \left(\frac{13x^3}{300} + \frac{x}{20}\right) = -0.34746$$

$$= \lim_{x \to 1.82} \left(\frac{13x^3}{300} + \frac{x}{20}\right) = 0.35224$$

4) *Subtracting lower limit from – higher limit:*

$$= 0.35224 - (-0.34746)$$

= 0.6997

5) *Finding area accurately:*

The Golden Gate bridge is also curved from the bottom. So, I will also have to model and equation for the lower arc and subtract its area. The equation of the bottom curve is $-0.01x^2 + 0x + 002$. I will now integrate it to find the area it covers.

$$\int_{-1.81}^{1.82} -0.01x^2 + 0.02dx$$

6) *Applying sum rule:*

$$= -\int 0.01x^2 dx + \int 0.02dx$$

7) *Solving the in-definite Integral:*

$$= 0.01 \int x^2\, dx = 0.01\frac{x^{2+1}}{2+1} = \frac{x^3}{300} + c$$

$$= 0.02 \int dx = \frac{x}{50}$$

$$= -\frac{x^3}{300} + \frac{x}{50} + c$$

8) *Adding boundaries:*

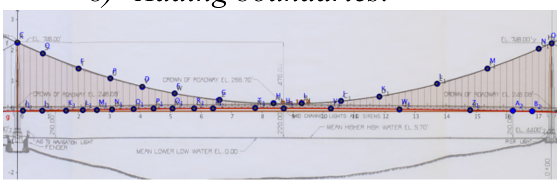


Image 10 – Appendix

$$= \lim_{x \to -1.81} \left(\frac{x}{50} - \frac{x^3}{300}\right) = -0.01643$$

$$= \lim_{x \to 1.82} \left(\frac{x}{50} - \frac{x^3}{300}\right) = 0.0163$$

9) Subtracting lower limit from – higher limit:

$= 0.0163 — 0.01643$
= 0.03273

Now, to find the exact area under the curve, I will subtract the bottom curves area from that of the top one. Therefore, 0.6997- 0.03273 = 0.66697.


Image 11 – Appendix

10) Finding Y coordinate of intersecting point:

Using GeoGebra, I was able to find the y coordinate; (0, 0.47). This is the point of intersection between the parabola and the integration boundaries.

11) Finding Areas:

I now have the integrated area, 0.66697. The area of the rectangle bounding the curve is 1.7061. It is the product of the range (3.63) (units between the lower and upper limit; -1.81 to 1.82) and the maximum point 0.47. Therefore, the area of the parabola would be the total area – the integrated area (1.7061– 0.66697 = 1.03913).

12) Finding Percentages:

$if\ 1.03913 \rightarrow 100$
$then\ 0.66697 \rightarrow ?$
Cross multiplying:
$\frac{0.66697 \times 100}{1.03913} = 60.18\%$

Table 5. Calculations for Graph of Bridge 2

(A)x^2	**From**	**To**	**Range**	**Integrated Area**	**Min Point**	**Total Area**	**Curve Area**	**Percentage**
0.13	-1.81	1.82	3.63	0.66697	0.47	1.7061	1.03913	64.18

Conclusion and Evaluation

It is now evident that my hypothesis was right. I was able to prove it through the examples of two real bridges. Throughout my calculations to understand the correlation between the two values, I got a constant answer in the ranges of 49-51. Finding a Pearson's Correlation Coefficient of above 0.7, proved how strongly these two values were correlated. Moving on to application in the real world, I made calculations on the two bridges that I had chosen. The result of my first bridge was 50.12% while the result of my second bridge was a little higher at 64.18%. Before, I predicted that the percentages would usually be close to 50%. My first result satisfies this criterion but the second bridge is off by about 14%. I think that this aims to prove that there is quite a lot of similarity in terms of the way such structures are built. I assume that this might be to provide structural stability to the bridges. Looking at it form a Physics point of view, I was able to understand that the arc bridges will never be a complete semi-circle. This is not possible because the action of forces makes the bridge quite instable. Therefore, this is why the bridge is never in the shape of a semi-circle but always smaller in order to make it stable. So, I assume, the ratio of 1:2 that I found, is a general proportion followed in such scenarios.

I think that there might have been a slight inaccuracy in the measurements because the image of the bridge that I had, was not exactly perpendicular; or a frontal view. This way, due to a slight angel, the graphs that I plotted on GeoGebra might have been slightly varied. They were analyzed in 2-Dimension and not the real world's, 3-Dimensions. Another issue was also that because these were digital images, they were quite pixelated. Placing my points accurately over the image so that I could find an equation for them was difficult because of this. I also did not take into consideration the thickness of the bridge's poles, beams and platforms. They were all considered as just as a single line to make calculations easier and accurate. Also, the horizontal length of the bridge was measured from the graph to make the measurements more proportionate and accurate as all other calculations were made from the image as well. While finding my correlation, I only used 4 values instead of the usually suggested 5. But this was important as I needed a balance between both positive and negative values. In terms of the strengths, I feel that the mathematical methods that I used to make calculations and, analyze the bridges were very accurate, appropriate and efficient. I tried my best to make no mistakes in my calculations and they are as concise as possible.

In terms of the limitations, such calculations can only be made on certain equations. One of the drawbacks is that, this method of calculation can only be applied to parabolic structures. Although, the calculations can be made on any equation no matter what the degree or limits are, this method is always applicable. This study might be useful to architects and engineers as it might help them understand the mathematics and physics behind such structures. Overall, this has been a really interesting exploration and I have learnt a lot and also look forward to projects such as this one!

Appendix 1

Image 1- Self. Screenshot. Digital image. N.p., n.d.

Image 2 - Self. Screenshot. Digital image. N.p., n.d.

Image 3 - Self. Screenshot. Digital image. N.p., n.d.

Image 4 - Self. Screenshot. Digital image. N.p., n.d.

Image 5 - Self. Screenshot. Digital image. N.p., n.d.

Image 6 - Self. Screenshot. Digital image. N.p., n.d.

Image 7 - Self. Screenshot. Digital image. N.p., n.d.

Image 8 - Self. Screenshot. Digital image. N.p., n.d.

Image 9 - Self. Screenshot. Digital image. N.p., n.d.

Image 10 - Self. Screenshot. Digital image. N.p., n.d.

Image 11 - Self. Screenshot. Digital image. N.p., n.d.

Bibliography

[1] "Integration." *Math First*. Massey University, 2003. Web. 19 Jan. 2016. <http://mathsfirst.massey.ac.nz/Calculus/integration/basics/definition.html>.

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[7] Chris Woodford. "Bridges and Tunnels." - *How Are They Built and How Do They Work?* N.p., 9 June 2015. Web. 13 Jan. 2016. <http://www.explainthatstuff.com/bridges.html>.

[8] "Passy's World of Mathematics." *Passys World of Mathematics*. N.p., 13 Oct. 2013. Web. 13 Jan. 2016. <http://passyworldofmathematics.com/lessons/page/13/>.